Vector Integration : Stokes's Theorem

P. Sam Johnson

National Institute of Technology Karnataka (NITK)
Surathkal, Mangalore, India

George Gabriel Stokes

Sir George Gabriel Stokes (1819-1903) was a physicist and mathematician.

Born in Ireland, Stokes spent all of his career at the University of Cambridge, where he served as Lucasian Professor of Mathematics from 1849 until his death in 1903.

George Gabriel Stokes

In physics, Stokes made seminal contributions to fluid dynamics (including the Navier-Stokes equations) and to physical optics.

In mathematics, he formulated the first version of what is now known as Stokes' theorem and contributed to the theory of asymptotic expansions.

His theoretical and experimental investigations covered hydrodynamics, elasticity, light, gravity, sound, heat, metrology, and solar physics.

He served as secretary, then president, of the Royal Society of London.

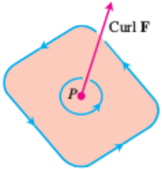
We have seen that the circulation density or curl of a two-dimensional field

$$\mathsf{F} = M\mathsf{i} + n\mathsf{j}$$

at a point (x, y) is described by the scalar quantity

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

In three dimensions, the circulation around a point P in a plane is described with a vector. This vector is normal to the plane of the circulation and points in the direction that gives it a right-hand relation to the circulation line.



Stokes's Theorem

The length of the vector gives the rate of the fluid's rotation, which usually varies as the circulation plane is tilted about P.

It turns out that the vector of greatest circulation in a flow with velocity field

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

is

$$\text{curl } \mathbf{F} = \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k}.$$

Stokes's theorem is the generalization to space of the circulation-curl form of Green's theorem.

Stokes's Theorem

Using the symbolic operator $\nabla = \mathrm{i}\frac{\partial}{\partial x} + \mathrm{j}\frac{\partial}{\partial y} + \mathrm{k}\frac{\partial}{\partial z}$, we have the formula for curl F.

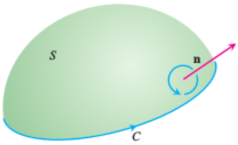
The symbol ∇ is pronounced "del".

The curl of F is $\nabla \times \mathrm{F}$:

$$\begin{aligned} \nabla \times \mathrm{F} &= \begin{pmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{pmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathrm{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathrm{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathrm{k} \\ &= \text{curl F}. \end{aligned}$$

Remember : curl $\mathrm{F} = \nabla \times \mathrm{F}$.

Stokes's theorem says that, under conditions normally met in practice, the circulation of a vector field around the boundary of an oriented surface of space in the direction counterclockwise with respect to the surface's unit normal vector field n equals the integral of the normal component of the curl of the field over the surface.



Stokes's Theorem

The circulation of $\mathrm{F} = M\mathrm{i} + N\mathrm{j} + P\mathrm{k}$ around the boundary C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector n equals the integral of $\nabla \times \mathrm{F.n}$ over S.

$$\int_C \mathrm{F}.d\mathrm{r} = \iint_S \nabla \times \mathrm{F.n}\ d\sigma. \tag{1}$$

Stokes's Theorem

Note that if two different oriented surfaces S_1 and S_2 have the same boundary C, then their curl integrals are equal:

$$\iint_{S_1} \nabla \times \mathsf{F}.\mathsf{n}_1 \; d\sigma = \iint_{S_2} \nabla \times \mathsf{F}.\mathsf{n}_2 \; d\sigma.$$

Both curl integrals equal the counterclockwise circulation integral on the left side of equation 1 as long as the unit normal vectors n_1 and n_2 correctly orient the surfaces.

Naturally, we need some mathematical restrictions on F, C, and S to ensure all the functions and derivatives involved be continuous.

Stokes's Theorem

If C is a curve in the xy-plane, oriented counterclockwise, and R is the region in the xy-plane bounded by C, then $d\sigma = dx\ dy$ and

$$(\nabla \times \mathrm{F}).\mathrm{n} = (\nabla \times \mathrm{F}).\mathrm{k} = \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big).$$

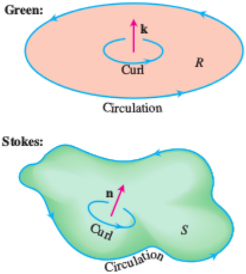
Under these conditions, Stokes's equation becomes

$$\int_C \mathrm{F}.d\mathrm{r} = \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\ dx\ dy$$

which is the circulation-curl from of the equation in Green's theorem.

Conversely, by reversing these steps we can rewrite the circulation-curl form of Green's theorem for two-dimensional fields in del notation as

$$\int_C \mathsf{F}.d\mathsf{r} = \iint_R \nabla \times \mathsf{F}.\mathsf{k}\ dA.$$



An Interpretation of $\nabla \times \mathrm{F}$

Suppose that $\mathrm{v}(x, y, z)$ is the velocity of a moving fluid whose density at (x, y, z) is $\delta(x, y, z)$ and let

$$\mathrm{F} = \delta \ \mathrm{v}.$$

Then $\int_C \mathrm{F}.d\mathrm{r}$ is the circulation of the fluid around the closed curve C.

By Stokes's theorem, the circulation is equal to the flux of $\nabla \times \mathrm{F}$ through a surface S spanning C :

$$\oint_C \mathrm{F}.d\mathrm{r} = \iint_R \nabla \times \mathrm{F}.\mathrm{n} \ d\sigma.$$

An Interpretation of $\nabla \times \mathrm{F}$

Suppose we fix a point Q in the domain of F and a direction u at Q.

Let C be a circle of radius ρ, with center at Q, whose plane is normal to u.

If $\nabla \times \mathrm{F}$ is continuous at Q, then the average value of the u-component of $\nabla \times \mathrm{F}$ over the circular disk S bounded by C approaches the u-compoent of $\nabla \times \mathrm{F}$ at Q as $\rho \to 0$:

$$(\nabla \times \mathrm{F}).\mathrm{u}_Q = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \iint_S \nabla \times \mathrm{F}.\mathrm{u} \; d\sigma.$$

If we replace the doube integral in the above equation by the circulation, we get

$$(\nabla \times \mathrm{F}).\mathrm{u}_Q = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \oint_C \mathrm{F}.d\mathrm{r}. \tag{2}$$

An Interpretation of $\nabla \times \mathbf{F}$

The left-hand side of the equation 2 has its maximum value when $\mathbf{u}$ is the direction of $\nabla \times \mathbf{F}$. When ρ is small, the limit on the right-hand of 2 is approximately

$$\frac{1}{\pi \rho^2} \oint_C \mathbf{F}\, d\mathbf{r},$$

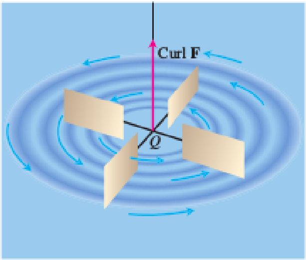
which is the circulation around C divided by the area of the disk (circulation density).

Suppose that a small paddle wheel or radius ρ is introduced into the fluid at Q, with its axle directed along $\mathbf{u}$.

The circulation of the fluid around C will affect the rate of spin of the paddle wheel.

An Interpretation of $\nabla \times \mathbf{F}$

The wheel will spin fastest when the circulation integral is maximized; therefore it will spin fastest when the axle of the paddle wheel points in the direction of $\nabla \times \mathbf{F}$.



An Important Identity

The following identity arises frequently in mathematics and the physical sciences.

For any function $f(x, y, z)$ whose second partial derivatives are continuous, we have

$$\text{curl grad } f = 0 \quad \text{or} \quad \nabla \times \nabla f = 0$$

because the mixed second derivatives are equal when the second partial derivatives are continuous.

Curl $\mathrm{F} = 0$ Related to the Closed-Loop Property

If $\nabla \times \mathrm{F} = 0$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D.

$$\oint_C \mathrm{F}.d\mathrm{r} = 0.$$

Exercises

Exercise

In the following exercise, use the surface integral in Strokes′ Theorem to calculate the circulation of the field F around the curve C in the indicated direction.

1. $F = x^2 i + 2xj + z^2 k$
   C : *The ellipse* $4x^2 + y^2 = 4$ *in the xy-plane, Counterclockwise when viewed from above*
2. $F = yi + xzj + x^2 k$
   C :*The boundary of the triangle cut from the plane* $x + y + z = 1$ *by the first octant, counterclockwise when viewed form above*

Solution for Exercise 1

1. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X^2 & 2X & Z^2 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (2-0)\mathbf{k} = 2\mathbf{k}$ and $\mathbf{n{=}k} \Rightarrow$ curl$\mathbf{F} \cdot \mathbf{n} =$ $dxdy \Rightarrow \oint_c \mathbf{F} \cdot d\mathbf{r} = \iint_R 2\ dA = 2 \Rightarrow d\sigma$ Area of the ellipse)$= 4\pi$

2. curl $\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\mathbf{i} - 2x\mathbf{j} + (z-1)\mathbf{k}$ and $\mathbf{n} = \frac{i+j+k}{\sqrt{3}} \Rightarrow curl\mathbf{f} \cdot \mathbf{n}$
$= \frac{1}{\sqrt{3}}(-x - 2x + z - 1) \Rightarrow d\sigma = \frac{\sqrt{3}}{1} dA \Rightarrow \oint_c \mathbf{F} \cdot d\mathbf{r} = \iint_R \frac{1}{\sqrt{3}}(-3x + z - 1)\sqrt{3} dA$
$= \int_0^1 \int_0^{1-x} [-3x + (1 - x - y) - 1] dy\ dx = \int_0^1 \int_0^{1-x} (-4x - y) dy\ dx =$
$\int_0^1 -[4x(1-x) + \frac{1}{2}(1-x)^2] dx$
$= -\int_0^1 \left(\frac{1}{2} + 3x - \frac{7}{2}x^2\right) dx = -\frac{5}{6}$

Exercises

Exercise

1. $\mathsf{F} = \left(y^2 + z^2\right)\mathsf{i} + \left(x^2 + y^2\right)\mathsf{j} + (x^2 + y^2)\mathsf{k}$
   C : The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy-plane, counter clockwise when viewed from above.
2. $\mathsf{F} = x^2y^3\mathsf{i} + \mathsf{j} + z\mathsf{k}$
   C :The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$, counterclockwise when viewed from above.

Solution for Exercise 2

1. curl
$$\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+y^2 & x^2+y^2 \end{vmatrix} = 2y\mathbf{i} + (2z-2x)\mathbf{j} + (2x-2y)\mathbf{k} \text{ and } \mathbf{n} = \mathbf{k}$$
$$\Rightarrow curl\mathbf{F}\cdot\mathbf{n} = 2x-2y \Rightarrow d\sigma = dx\ dy \Rightarrow \oint_c \mathbf{f}\cdot dr = \int_{-1}^{1}\int_{-1}^{1}(2x-2y)dx\ dy =$$
$$\int_{-1}^{1}[x^2-2xy]_{-1}^{1}dy$$
$$= \int_{-1}^{1} -4y\ dy = 0$$

2. curl
$$\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - 3x^2y^2\mathbf{k} \text{ and } \mathbf{n} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{4}$$
$$\Rightarrow \text{ curl } \mathbf{F}\cdot\mathbf{n} = -\tfrac{3}{4}x^2y^2z; d\sigma = \tfrac{4}{z}dA$$
$$\Rightarrow \oint_c \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(-\tfrac{3}{4}x^2y^2z\right)\left(\tfrac{4}{z}\right) dA$$
$$= -3\int_0^{2\pi}\int_0^2 (r^2\cos^2\theta)(r^2\sin^2\theta)r\ dr\ d\theta = -3\int_0^{2\pi}\left[\tfrac{r^6}{6}\right]_0^2 (\cos\theta\sin\theta)^2 d\theta =$$
$$-32\int_0^{2\pi}\tfrac{1}{4}\sin^2 2\theta d\theta = -4\int^{4\pi} 0\sin^2 u\ du$$
$$= -4\left[\tfrac{u}{2} - \tfrac{\sin 2u}{4}\right]_0^{4\pi} = -8\pi$$

Exercise

1. *Let* n *be the outer unit normal of the elliptical shell* $S; \quad 4x^2 + 9y^2 + 36z^2 = 36, \quad z \geq 0$ *and let* $\mathrm{F} = y\mathrm{i} + x^2\mathrm{j} + \left(x^2 + y^4\right)^{3/2} \sin e^{\sqrt{xyz}}\mathrm{k}.$
   Find the value of $\displaystyle\iint_S \nabla \times \mathrm{F} \cdot n \, d\sigma$
   (Hint: One parametrization of the ellipse at the base of the shell is $x = 3\cos t, y = 2\sin t, 0 \leq t \leq 2\pi.$*).*
2. *Let* n *be the outer unit normal (normal away from the origin) of the parabolic shell* $S: \quad 4x^2 + y + z^2 = 4, \quad y \geq 0,$ *and let* $\mathrm{F} = \left(-z + \frac{1}{2+x}\right)\mathrm{i} + \left(\tan^{-1} y\right)\mathrm{j} + \left(x + \frac{1}{4+z}\right)\mathrm{k}.$
   Find the value of $\displaystyle\iint_S \nabla \times \mathrm{F} \cdot n \, d\sigma$

Solution for Exercise 3

1. $x = 3\cos t$ and
$y = 2\sin t \Rightarrow \mathbf{F} = (2\sin t)\mathbf{i} + (9\cos^2 t)\mathbf{j} + (9\cos^2 t + 16\sin^4 t)\sin e^{\sqrt{(6\sin t \ \ \cos \ \ t)(0)}}\mathbf{k}$ at the base of the shell: $\mathbf{r} = (3\cos t)\mathbf{i} + (2\sin t)\mathbf{j} \Rightarrow d\mathbf{r} = (-3\sin t)\mathbf{i} + (2\cos t)\mathbf{j} \Rightarrow \mathbf{F}\cdot\frac{dr}{dt} =$
$-6\sin^2 t + 18\cos^3 t \Rightarrow \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}d\sigma = \int_0^{2\pi}(-6\sin^2 t + 18\cos^3 t)dt =$
$[-3t + \frac{3}{2}\sin 2t + 6(\sin t)(\cos^2 t + 2)]_0^{2\pi} = -6\pi$

2. curl $\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z + \frac{1}{2+x} & tan^{-1}y & x + \frac{1}{4+z} \end{vmatrix} = -2\mathbf{j};\ f(x,y,z) = 4x^2 + y + z^2 \Rightarrow$
$\nabla f = 8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k} \Rightarrow \mathbf{n} = \frac{\nabla f}{|\nabla f|}$ and
$\mathbf{p} = \mathbf{j} \Rightarrow |\nabla f\cdot\mathbf{p}| = 1 \Rightarrow d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}dA = |\nabla f|dA;\ \nabla\times\mathbf{F}\cdot\mathbf{n} = \frac{1}{|\nabla f|}(-2\mathbf{j}\cdot\nabla f) = \frac{-2}{\nabla f}$
$\Rightarrow \nabla\times\mathbf{F}\cdot\mathbf{n}d\sigma = -2dA \Rightarrow \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}d\sigma = \iint_R -2dA = -2($ Area of $R) =$
$-2(\pi\cdot 1\cdot 2) = -4\pi$, where R is the elliptic region in the $xz-$ plane enclosed by $4x^2 + z^2 = 4$.

Exercise

1. *Let S be the cylinder $x^2 + y^2 = a^2, 0 \leq z \leq h$, together with its top, $x^2 + y^2 \leq a^2, z = h$. Let* $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. *Use Stokes ′ Theorem to find the flux of* $\nabla \times \mathbf{F}$ *outward through S.*
2. Flux of Curl F. *Show that* $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$ *has the same value for all oriented surfaces S that span C and that induce the same positive direction on C.*
3. *Let F be a differentiable vector field defined on a region containing a smooth closed oriented surface S and its interior. Let* $\mathbf{n}$ *be the unit normal vector field on S. Suppose that S is the union of two surfaces S_1 and S_2 joined along a smooth simple closed curve C. Can anything be said about* $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$*?*
   Give reasons for your answer.

Solution for Exercise 4

1. Flux of $\nabla \times \mathbf{F} = \int_S \int \nabla\nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \oint_c \mathbf{F} \cdot d\mathbf{r}$, so let C be parametrized by $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, 0 \leq t \leq 2\pi \Rightarrow \frac{dr}{dt} = (-a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} \Rightarrow \mathbf{F} \cdot \frac{dr}{dt} = ay \ \sin t + ax \ \cos t = a^2 \sin^2 t + a^2 \cos^2 t = a^2 \Rightarrow$ Flux of $\nabla \times \mathbf{F} = \oint_c \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} a^2 dt = 2\pi a^2$
2. Let S_1 and S_2 be oriented surfaces that span C and that induce the same positive direction on C. Then $\int\int_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}_1 d\sigma_1 = \oint_c \mathbf{F} \cdot d\mathbf{r} = \int\int_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}_2 d\sigma_2$
3. $\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \int\int_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma + \int\int_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$, and since S_1 and S_2 are joined by the simple closed curve C, each of the above integrals will be equal to a circulation integral on C. But for one surface the circulation will be counterclockwise, and for the other surface the circulation will be clockwise. Sinc ethe integrands are the same, the sum will be $0 \Rightarrow \int_X \int \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = 0$.

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